Math 2010E

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Lec MWF 1:30–3:15 (50 + 5 + 50)

Tut M 3:30 – 4:15

F 12:30 – 1:15

5 assignments			10%
6/10 (F)	midterm	(tentative)	35%
6/29 (W)	final		55%

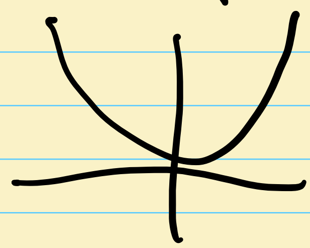
Differential calculus of <u>multi-variable</u> functions

<u>Recall</u> one-variable calculus

$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2$

differentiation $f'(x) = 2x$. $f''(x) = 2$.

· graph

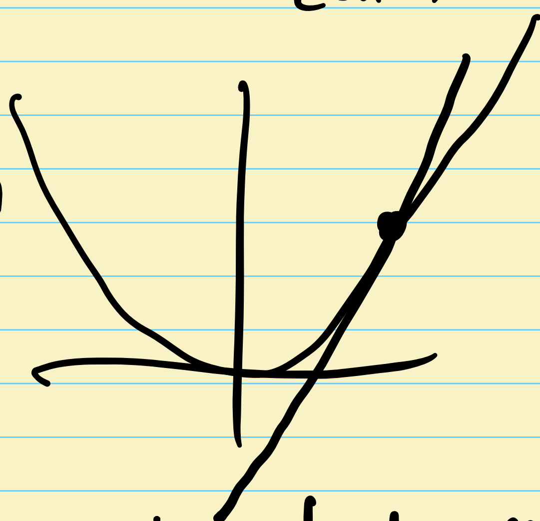


· f', f'' (meaning, velocity, acceleration)

convex concave.

· min max

· apporximation



<u>Our goal</u>. Deal with vector-valued multi-variable functions

$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x, y) = x^2 + y^2$

$g : \mathbb{R}^3 \to \mathbb{R}^2 \qquad g(x, y, z) = (z \sin(xy), y + z)$

- graph?
- differentiation? (rate of change)
- min, max

⊛ approximation.

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

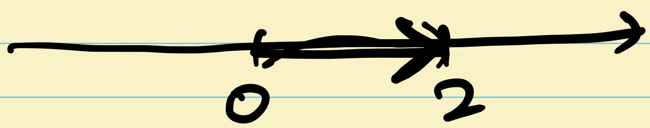
Euclidean space $\mathbb{R}^n$.

$$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \quad (n\text{-copies of } \mathbb{R})$$

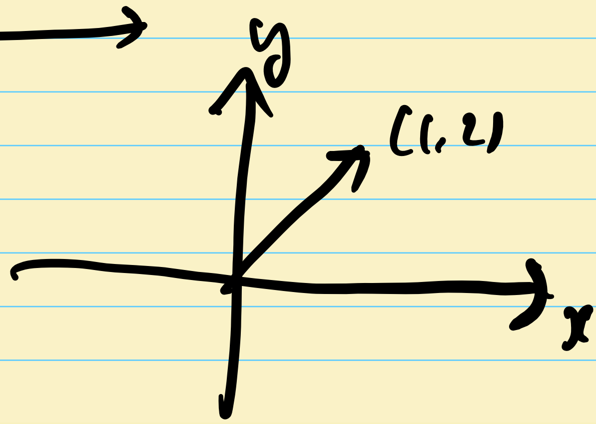
$$= \{ (x_1, \cdots, x_n) \mid x_i \in \mathbb{R},\ 1 \le i \le n \}$$

its elements are called n-dimensional <u>vectors</u>

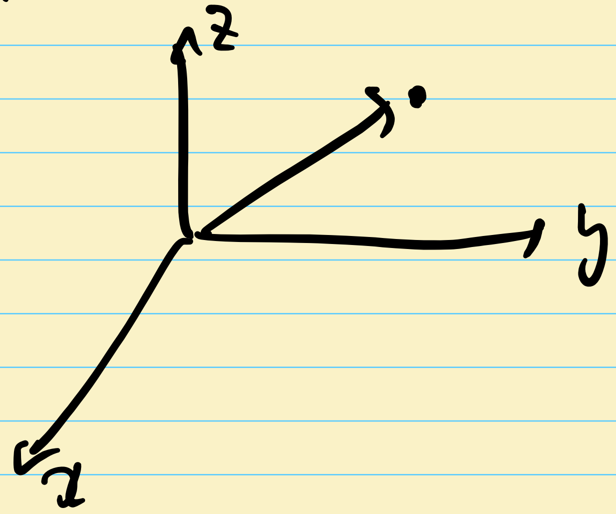
<u>Ex</u> ① $\mathbb{R} \ni 2$



② $\mathbb{R}^2 \ni (1,2)$



③ $\mathbb{R}^3 \ni (1,1,1)$



$\mathbb{R}^4$ or higher $\mathbb{R}^n$: difficult to draw.
but deal with them.

Rmk

- Each $(x_1, \cdots, x_n) \in \mathbb{R}^n$ can be viewed as a point or <u>a vector</u> in $\mathbb{R}^n$.
- denoted by $\vec{v}$, $\vec{u}$, $\vec{w}$, $\vec{a}$ ...
- the vector $(0, \cdots, 0)$ is called the zero vector. denoted by $\vec{0}$.

Basic operations of vectors

$$\vec{v} = (v_1, \cdots, v_n) \quad \in \mathbb{R}^n$$
$$\vec{w} = (w_1, \cdots, w_n)$$

$r \in \mathbb{R}$ a real number.

Def

· (addition) ✓

$$\vec{v} + \vec{w} = (v_1 + w_1, \cdots, v_n + w_n)$$

real # r

· (scalar multiplication) ✓

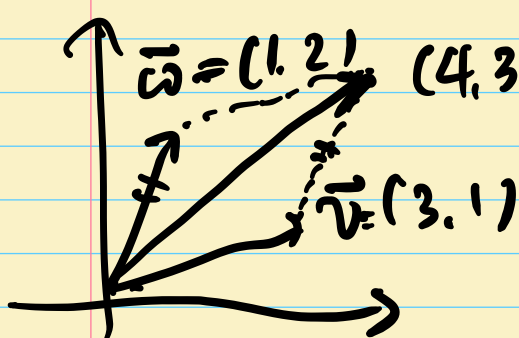
$$r\vec{v} = (rv_1, \cdots, rv_n)$$

· (subtraction)

$$\vec{v} - \vec{w} = (v_1 - w_1, \cdots, v_n - w_n)$$
$$(= \vec{v} + (-1)\vec{w})$$

Geometric interpretation.

(addition) $\vec{v} = (3,1)$ $\quad \vec{w} = (1,2)$ $\quad \in \mathbb{R}^2$



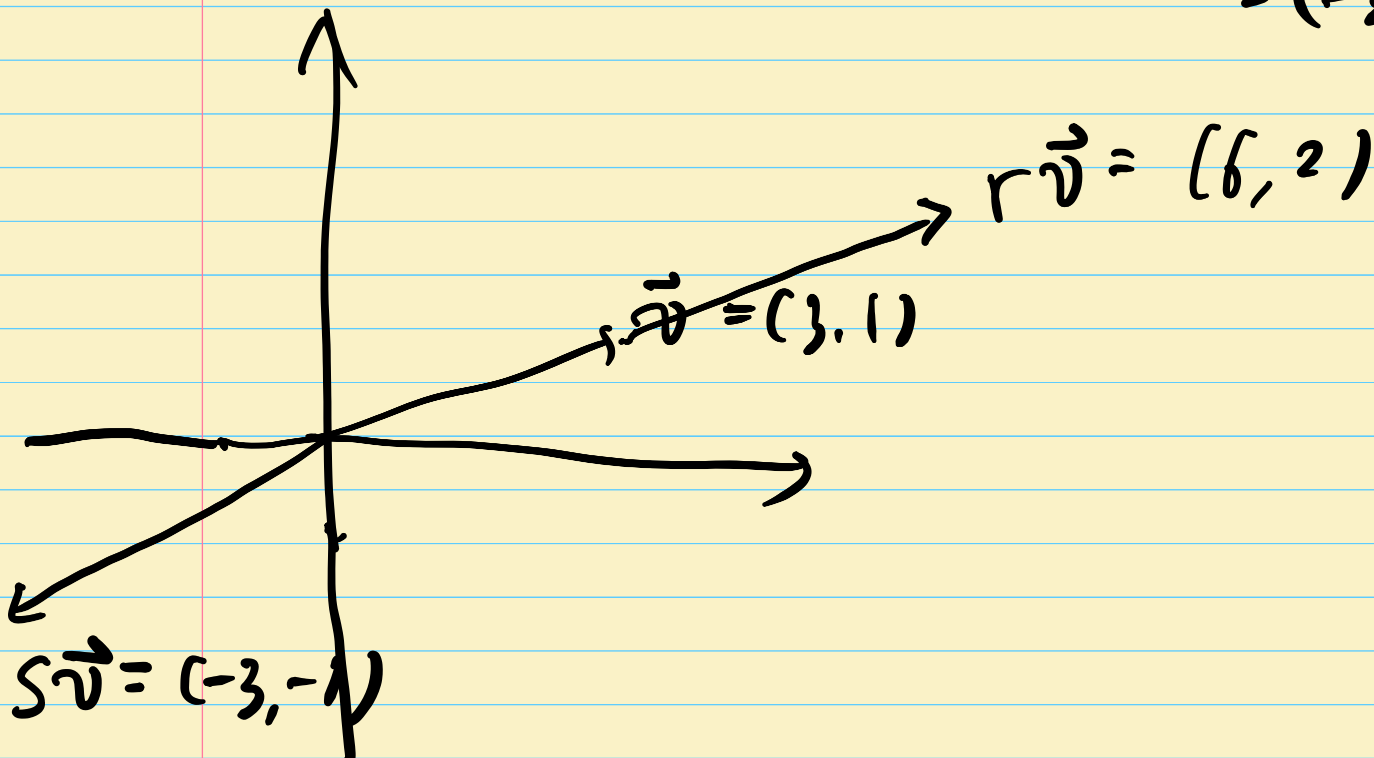
$$(4,3) = \vec{v} + \vec{w} = (3+1, 1+2)$$
$$= (4,3)$$

(scalar multiplication)

$\vec{v} = (3,1)$. $r = 2$. $s = -1$

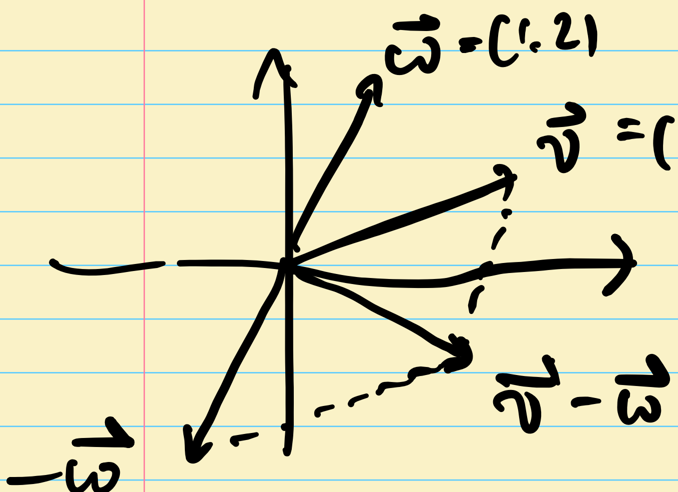
$r\vec{v} = (6,2)$ $s\vec{v} = -\vec{v}$

$= (-3,-1)$



(subtraction)

$\vec{v} = (3,1)$ $\vec{w} = (1,2)$



$\vec{v} - \vec{w}$

$= (2,-1)$

Prop $\vec{u}, \vec{v}, \vec{w}$ are vectors

$\alpha . \beta \in \mathbb{R}$

① $0 \cdot \vec{v} = \vec{0}$

② $1 \cdot \vec{v} = \vec{v}$

③ (associativity) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

④ (commutativity) $\vec{v}+\vec{w} = \vec{w}+\vec{v}$

⑤ $\vec{v}+\vec{0} = \vec{v}$

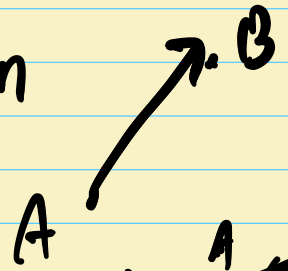
⑥ (distributivity) $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$

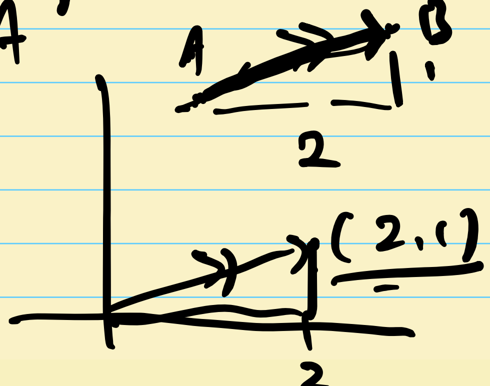
⑦ (") $\alpha(\vec{v}+\vec{w}) = \alpha\vec{v}+\alpha\vec{w}$

⑧ $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$

(proof) easy.

A, B two points in $\mathbb{R}^n$

An  is also called a vector denoted $\overrightarrow{AB}$.



$A = (x_1, x_2)$

$B = (y_1, y_2)$

$\vec{AB}$ (in $\mathbb{R}^2$) $= (y_1 - x_1,\ y_2 - x_2)$

$\mathbb{R}^n$; $A = (x_1 \cdots x_n)$

$B = (y_1 \cdots y_n)$

$\vec{AB} = (y_1 - x_1, \dots, y_n - x_n)$

<u>Ex</u> $A = (1,0)$, $B = (3,3)$, $C = (2,4)$, $D = (0,1)$

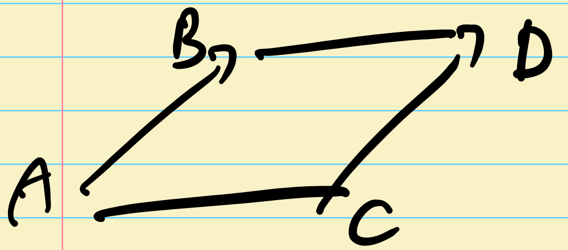
ABCD is a parallelogram.

(pf) $\vec{AB} = \vec{DC}$?

$\vec{AB} = (3,3) - (1,0) = (2,3)$

$\vec{DC} = (2,4) - (0,1) = (2,3)$

$\vec{AB} = \vec{CD}$



ABCD is a parallelogram. $\square$

Length & dot product.

<u>Def</u> The length (norm, magnitude) of a vector $\vec{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

<u>Def</u> The dot product of two vectors $\vec{v} = (v_1 \cdots v_n)$, $\vec{w} = (w_1 \cdots w_n)$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$$

<u>Prop</u> $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $r \in \mathbb{R}$ Then

1. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
   $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
2. $(r\vec{v}) \cdot \vec{w} = r(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (r\vec{w})$
3. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ ✓
5. $\|r\vec{v}\| = |r| \, \|\vec{v}\|$
6. $\vec{v} \cdot \vec{v} \geq 0$ and $\vec{v} \cdot \vec{v} = 0$ iff $\vec{v} = \vec{0}$
7. $(n = 2, 3)$ $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$ ✓

   θ is the angle between $\vec{v}$ and $\vec{w}$.

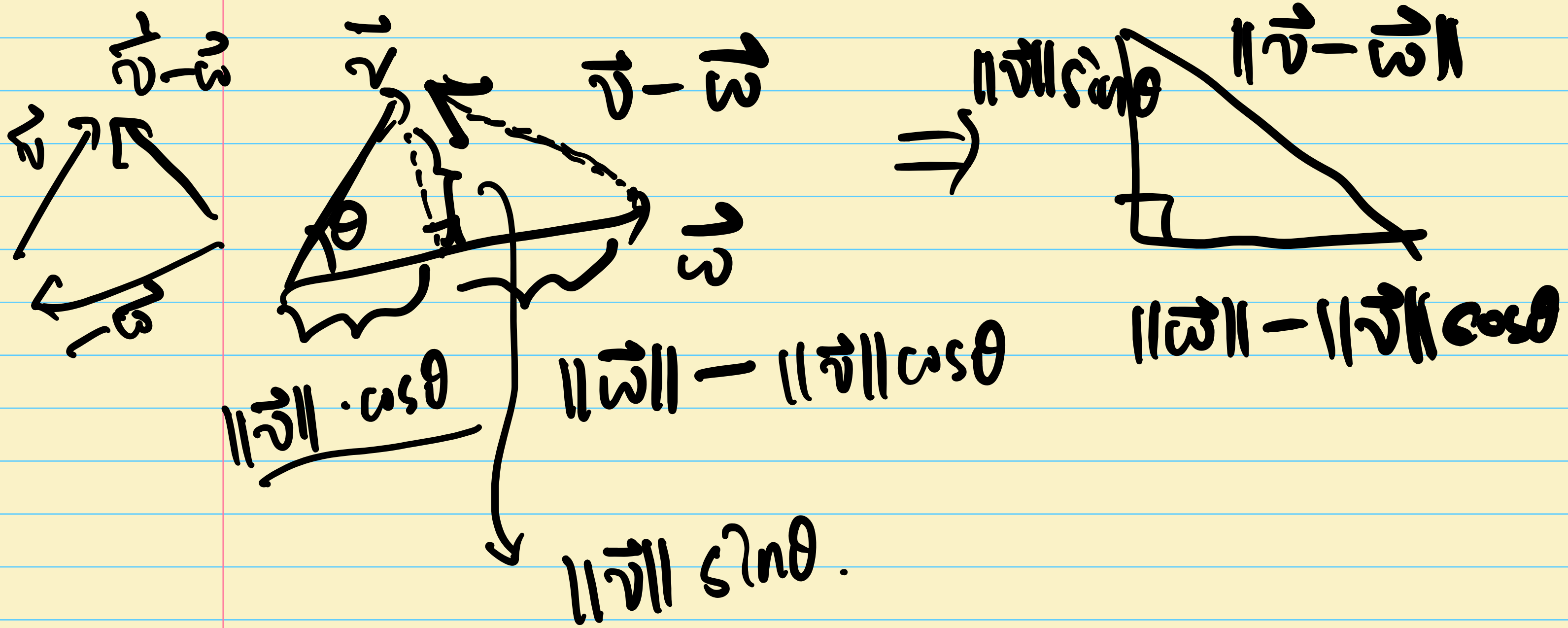
In ptc, if $\vec{v}, \vec{w} \neq \vec{0}$, then

$\vec{v} \cdot \vec{w} = 0$ iff $\cos\theta = 0$ $\left(\theta = \frac{\pi}{2}\right)$

iff $\vec{v} \perp \vec{w}$

(proof of ⑦)

For simplicity, $\theta < \frac{\pi}{2}$



$$\Rightarrow \underline{\|\vec{v}-\vec{w}\|^2} = (\|\vec{v}\|\sin\theta)^2 + (\|\vec{w}\| - \|\vec{v}\|\cos\theta)^2$$

$$= \|\vec{v}\|^2\sin^2\theta + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta + \|\vec{v}\|^2\cos^2\theta$$

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$$ ①

On the other hand

$$\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$

$$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$ ②

①, ② $\Rightarrow$ $\cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2} - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$

$= \cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2} - 2\vec{v}\cdot\vec{w}$

$\therefore \vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ $\square$

Similar for $\theta \geq \frac{\pi}{2}$

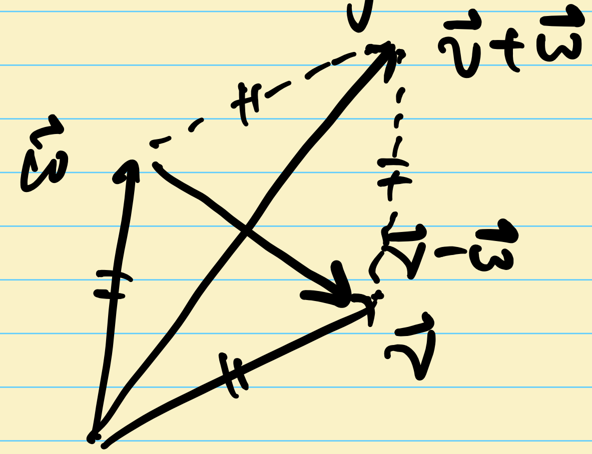
Rmk · A vector of length 1 is called a unit vector.

· $\vec{v} \neq \vec{0}$, then $\frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector.

Ex $\|\vec{v}\| = \|\vec{w}\|$ Then show that
$$(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = 0.$$

(geometric meaning)

($\mathbb{R}^2$)



The diagonals of rhombus are perpendicular.

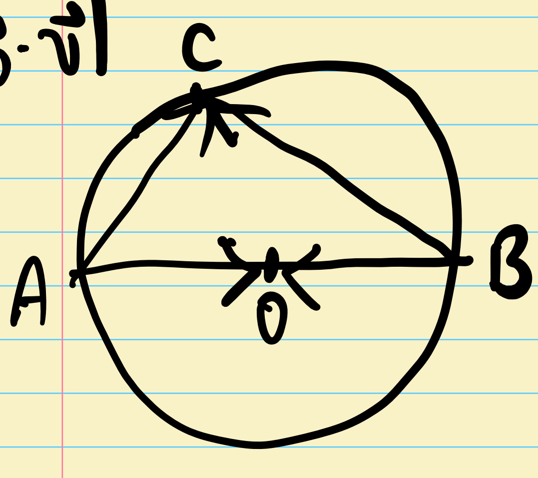
(proof) $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w})$

$$= \underbrace{\vec{v}\cdot\vec{v}} - \cancel{\vec{v}\cdot\vec{w}} + \cancel{\vec{w}\cdot\vec{v}} - \vec{w}\cdot\vec{w}$$

$$= \|\vec{v}\|^2 - \|\vec{w}\|^2$$

$\left(\begin{array}{l}\vec{v}\cdot\vec{w}\\ = \vec{w}\cdot\vec{v}\end{array}\right)$ $= 0.$ $\square$

Ex



$\angle ACB = \frac{\pi}{2}.$

(pf) want: $\vec{AC}$ and $\vec{BC}$ are perpendicular.

$\Leftrightarrow \vec{AC} \cdot \vec{BC} = 0.$

$$\vec{AC} = \vec{AO} + \vec{OC}$$

$$\vec{BC} = \vec{BO} + \vec{OC} = -\vec{AO} + \vec{OC}$$

$$\vec{AC} \cdot \vec{BC} = (\vec{AO} + \vec{OC}) \cdot (\vec{BO} + \vec{OC})$$

$$= (\vec{AO} + \vec{OC}) \cdot (-\vec{AO} + \vec{OC})$$

$$= -\vec{AO} \cdot \vec{AO} + \cancel{\vec{AO} \cdot \vec{OC}} - \cancel{\vec{OC} \cdot \vec{AO}} + \vec{OC} \cdot \vec{OC}$$

$$= -\|\vec{AO}\|^2 + \|\vec{OC}\|^2$$

$$= 0 \qquad \left(\because \|\vec{AO}\| = \|\vec{OC}\| = \text{radius of circle} \right)$$

$\square$

Two inequalities

Thm (Cauchy–Schwarz inequality)

For all $\vec{a}, \vec{b} \in \mathbb{R}^n$

$$\|\vec{a} \cdot \vec{b}\| \leq \|\vec{a}\| \cdot \|\vec{b}\|$$

norm of vectors

absolute value of a real number.

dot product of vectors

multiplication of real numbers

Rmk If $n = 2, 3$

$$\|\vec{a} \cdot \vec{b}\| = \| \|\vec{a}\| \|\vec{b}\| \cos\theta \|$$

$$= \|\vec{a}\| \|\vec{b}\| \, |\cos\theta|$$

$$\leq \|\vec{a}\| \|\vec{b}\| .$$

(pf)

$$0 \leq \|\vec{a} - t\vec{b}\|^2 \qquad t \in \mathbb{R} \text{ any real \#.}$$

$$= (\vec{a} - t\vec{b}) \cdot (\vec{a} - t\vec{b})$$

$$= \underbrace{\|\vec{a}\|^2 - 2t(\vec{a} \cdot \vec{b}) + \|\vec{b}\|^2 t^2}_{\text{a quadratic function on } t}$$

∴ its discriminant must be non-negative.

$$\therefore \quad 4(\vec{a}\cdot\vec{b})^2 - 4\|\vec{a}\|^2\|\vec{b}\|^2 \leq 0.$$

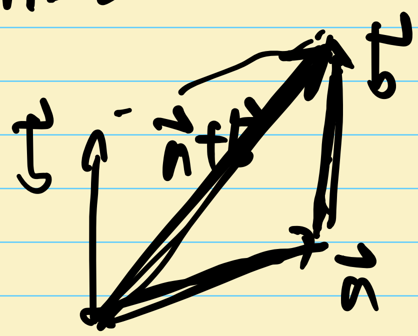
$$\therefore \quad \|\vec{a}\cdot\vec{b}\| \leq \|\vec{a}\|\|\vec{b}\| \qquad \square$$

Thm (triangle inequality)

For any $\vec{a}, \vec{b} \in \mathbb{R}^n$,

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

Rmk $n=2$



(proof) $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$$= \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$$= \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$\vec{a}\cdot\vec{b} \leq \|\vec{a}\|\|\vec{b}\|$ (∵ C-S inequality)

$$\leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$$

$$= (\|\vec{a}\| + \|\vec{b}\|)^2$$

i.e. $\|\vec{a}+\vec{b}\|^2 \leq (\|\vec{a}\| + \|\vec{b}\|)^2$

$\therefore \|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\| \quad \square$

C-S $\Rightarrow$ triangle

<u>Exercise</u> $\Leftarrow$

scaler product $r \cdot \vec{v}$: vector

dot product $\vec{v} \cdot \vec{w}$: scalar.

Cross product

$\vec{v} \times \vec{w}$: a vector.

※ in $\mathbb{R}^3$

Recall determinant of a matrix

2×2, 3×3.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

✓

<u>notation</u> $\hat{i} = (1, 0, 0)$

$\hat{j} = (0, 1, 0) \quad \in \mathbb{R}^3$

$\hat{k} = (0, 0, 1)$

standard unit vectors.

<u>Def</u> (cross product)

$\vec{a} = (a_1, a_2, a_3) \quad \in \mathbb{R}^3$

$\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

$$\in \mathbb{R}^3$$

$\underline{\text{Ex}}$ ① $\hat{i} \times \hat{j} = (1,0,0) \times (0,1,0)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} + 1\hat{k}$$

$$= \hat{k}$$

② $\vec{a} = (2.\ 3.\ 5)$

$\vec{b} = (1.\ 2.\ 3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -\hat{i} - \hat{j} + \hat{k}$$

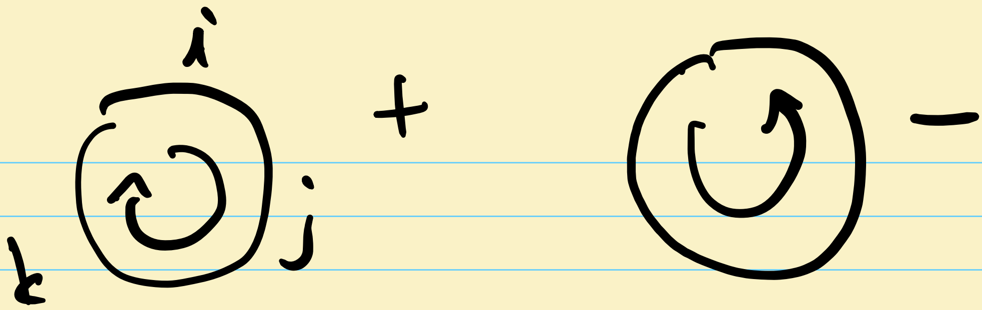
$$= (-1.\ -1.\ 1)$$

<u>Rmk</u>

$\hat{i} \times \hat{i} = \vec{0}$ $\quad \hat{i} \times \hat{j} = \hat{k}$ $\quad \hat{i} \times \hat{k} = -\hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$ $\quad \hat{j} \times \hat{j} = \vec{0}$ $\quad \hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$ $\quad \hat{k} \times \hat{j} = -\hat{i}$ $\quad \hat{k} \times \hat{k} = \vec{0}$



Properties of cross product.

Prop $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

① $\vec{a} \times \vec{a} = 0$

② $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (Recall $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$)

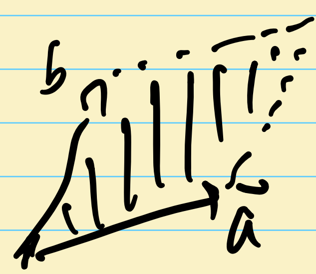
③ $(\alpha\vec{a} + \beta\vec{b}) \times \vec{c}$

$= \alpha\vec{a} \times \vec{c} + \beta\vec{b} \times \vec{c}$

④ θ : angle between $\vec{a}$ and $\vec{b}$.

$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \underline{\sin\theta}$

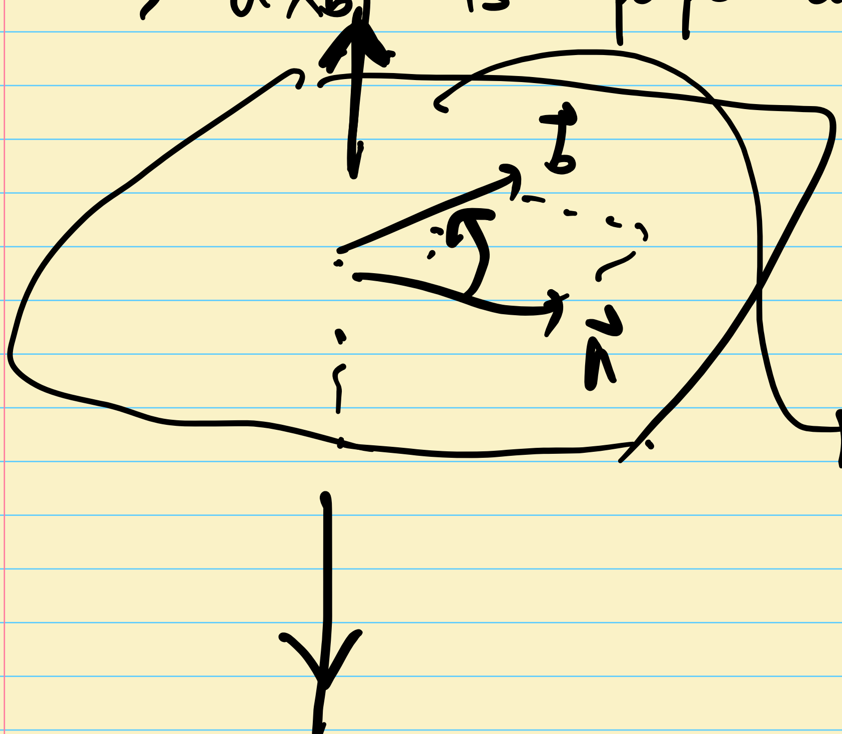
= area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$



⑤ $(\vec{a}\times\vec{b})\cdot\vec{a} = (\vec{a}\times\vec{b})\cdot\vec{b} = 0$

Rmk ⑤; $(\vec{a}\times\vec{b})\cdot\vec{a} = (\vec{a}\times\vec{b})\cdot\vec{b} = 0$.

$\Rightarrow \vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$ & $\vec{b}$.

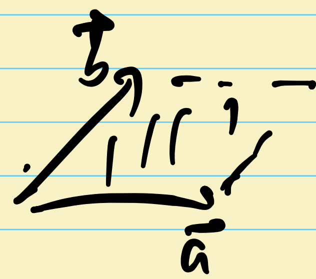


which direction?

right-hand rule

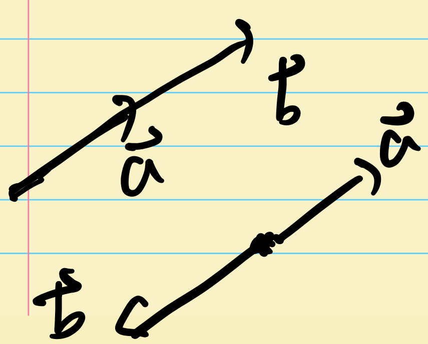
direction of $\vec{a}\times\vec{b}$.

$\|\vec{a}\times\vec{b}\|$ length? area of



④; $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow$ area of parallelogram $=0$

$\Leftrightarrow \vec{a}, \vec{b}$ lies on the same line



(proof) ①, ②, ③, ⑤ straight forward.

④ ; $\|\vec{a}\times\vec{b}\|^2$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta$$

$\left(\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta\right)$

$$= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta)$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta.$$

$0\le\theta\le\pi \rightsquigarrow \sin\theta\ge 0$

$$\therefore \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta. \quad \square$$

Ex $A=(1,2,1)$, $B=(1,-1,0)$, $C=(2,3,2)$

consider a plane P by ABC

Q Find a normal vector of P

(vector perpendicular to P) ?